SETS

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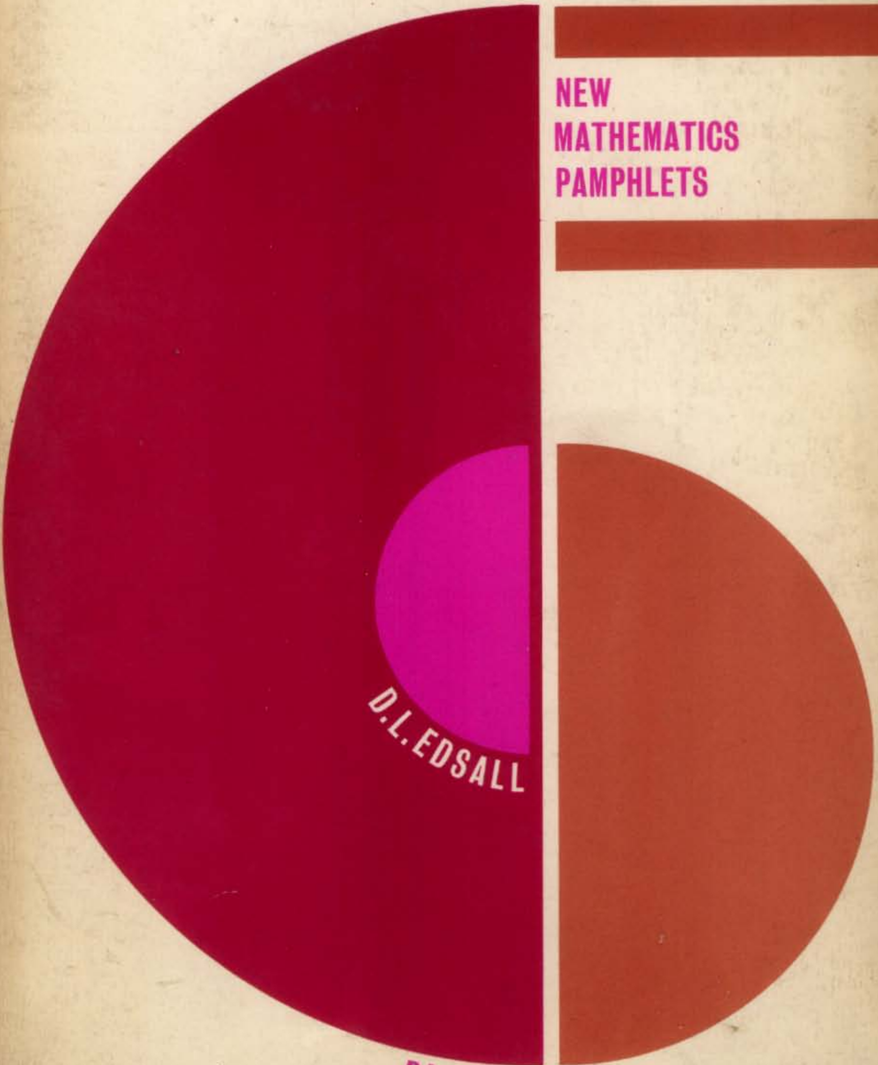

D.L. EDSALL

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NEW MATHEMATICS PAMPHLETS

SETS

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VECTORS AND MATRICES IN TWO DIMENSIONS

GRAPHS AND INEQUALITIES

NEW MATHEMATICS PAMPHLETS

SETS

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Introduction

I feel that the importance of Set Theory in the junior school curriculum, is that it offers a basic language that recurs later in many branches of mathematics. At the same time it represents a truly mathematical approach—starting with a basic set of definitions, and deducing from these sundry results. Moreover it can be studied at that stage of the pupil's development when he or she thinks that mathematics is almost entirely number-bound, and its use of letters and symbols to represent ideas rather than numbers is a good illustration of the width of application of the subject.

It would, though, be a great mistake, to think that this subject should be studied in isolation, and its reappearance in connection with graphs, probabilities, geometry, and logic theory should all be stressed when these topics are ready for discussion. This pamphlet tries to supply sufficient detail to act as a basic language for later use with subjects such as these. I have tried to make the text readable for the pupils, so that when they want to refresh their ideas on a certain topic, the text will provide sufficient information in a language that they can understand.

I would like to thank Mr. R. M. Dunnett for the assistance that he has given me in checking the exercises and the proof reading of the text.

SETS

You are probably arranged in school sets for learning certain subjects such as Mathematics or French, or you may own a "geometry" set consisting of a pair of compasses, a protractor, and so on, and you will therefore understand the idea of a *set* as a collection of people or things, all of which have some property in common. Your mathematics set is a collection of pupils, all of whom have about the same mathematical ability; the "geometry set" is a collection of instruments which are useful for drawing geometrical figures. Both these rules are rather vague, but in general terms they define the collection of objects which make up the set. The kinds of sets in which we shall be interested are also defined by means of rules, but because we want to make more accurate deductions from our sets we must be rather more precise in our defining rules.

Definition 1. A *set* is a collection of objects such that when each one of these objects is placed against the rule for the set, then that rule is truly satisfied.

Definition 2. Each object that obeys the rule for a set is called an *element* of that set.

Thus there can be no ambiguous cases—the rule must be such that when any object is put alongside it, the answer is "Yes, it has this property" (i.e. it is an element of the set) or "No, it has not". If there are any ambiguous objects which may or may not have the property, the rule does not define a set. Thus "all positive even numbers less than 20" defines the set consisting of the numbers 2, 4, 6, 8, 10, 12, 14, 16, and 18—but "all those men playing cricket for England in 1990" does not define a set, for we cannot tell as yet who will play. Notice that sets are collections of objects which need not necessarily be numbers. All we require is a clearly defined rule for the set; the kind of object which it contains is irrelevant.

EXERCISE 1

1. Which of the following rules define a set?

(*a*) All positive numbers less than 15.

(*b*) All cars with registration numbers beginning LTD.

(*c*) All those pupils in the school who will have passed Ordinary Level Elementary Mathematics by next January.

(*d*) All pupils in the school who cannot swim one length of a swimming bath.

(*e*) All schools which have blue and yellow as their school colours.

(*f*) All pupils in the school who are good at mathematics.

Just as in algebra we use letters to stand for numbers, so with sets we use capital letters to stand for sets. For example, let A stand for the set of all positive whole numbers less than 15. This gives us one way of explaining what we mean by the set A —a description in words; but clearly the words "is the set of all" are likely to appear in any such definition, and so it is more

usual to write this statement in a symbolic way as

$$A \text{ is } \{X \mid X \text{ is a positive whole number} < 15\}$$

Here the curly brackets imply "the set of all",

$|$ means "such that"

$<$ means "less than"

$>$ means "greater than"

(These symbols are all listed on p. 38.) The expression therefore reads fully as "A is the set of all X such that X is a positive whole number less than 15".

An alternative method of defining a set is to list all the elements of the set. In this way A could be written as

$$\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14\}$$

Whilst this is convenient for small sets, it clearly gets cumbersome when there is a large number of elements in the set, or when the name of each element is long. But some sets would be difficult to detail in any other way; for example, consider the set {banana, teaspoon, mug}. (If you think that these objects are not connected by any rule, the actual listing of the objects makes up the rule. Check that Definition 1 is satisfied —does an apple belong to the set?)

Small letters can also be used to represent individual elements of a set. But notice that there is a difference between the set $\{a, b, c, d\}$ which consists of just those elements, and saying that a is one of the elements of a set A; the second case means we have taken any element of A, and to show that it belongs to the set A we have called it a. If we were to take several elements from the set A we could label them with suffices as a_1, a_2, a_3, and so on.

EXERCISE 2

1. Write down all the elements of the following sets:

A is $\{x \mid x \text{ is a whole number} < 9\}$
B is $\{x \mid x \text{ is a pupil in your mathematics set}\}$
C is $\{x \mid x \text{ satisfies the equation } 3x+5 = 1\}$
D is $\{x \mid x \text{ satisfies the equation } x^2 = 9\}$
E is $\{x \mid x \text{ is any arrangement of the letters } pqr\}$
F is $\{x \mid x^2 < x < 10, x \text{ a whole number}\}$

2. Write down the following sets using the symbolic method:

The set of all whole numbers greater than 17.
The set of all pupils in the school who are under 15.
The set of all pupils who are members of your form.
The set of all numbers which are factors of 48.
The set of all members of your form who are over 20 years old.
The set of all numbers which satisfy the equation $x^2+1 = 0$.

3. There is something specially interesting about the last two examples in question 2, which makes them different from other examples. Can you see what it is? Write down another set which has the same property.

4. Draw diagrams to illustrate all the positions of the points which form the elements of the following sets (such sets are called *loci*):

(i) The set of all points which are in a plane and which are equidistant from a fixed point.

(ii) $\{\mathrm{P} \mid \mathrm{PA} = \mathrm{PB}$, where A and B are two fixed points, and PAB all lie in a plane$\}$.

(iii) $\{\mathrm{P} \mid \mathrm{PA}^2+\mathrm{PB}^2 = \mathrm{AB}^2$, where A and B are two fixed points, and PAB all lie in a plane$\}$.

(iv) The set of all the centres of circles which lie in a plane and pass through two fixed points A and B.

(v) The set of all the centres of circles which lie in a plane and pass through a fixed point and all have the same radius.

5. What happens to the sets in question 4 if the points in question are not restricted to lying in a plane?

6. Draw accurate diagrams to illustrate the lines which are the elements of the following sets (these lines are said to form an *envelope*; notice how the shape of the curve is clearly shown without any curved lines being actually drawn):

(i) The set of all lines PQ, where P lies on a fixed line OX, Q lies on a fixed line OY, OX is perpendicular to OY, and PQ is of constant length.

(ii) The set of all lines PQ, where P lies on a fixed line OX, and Q lies on a fixed line OY, and the line moves so that when P moves a distance x towards O, then Q moves the same distance away from O.

(iii) P is a fixed point, and AB a fixed line. The set is that set of all lines XY such that X lies on AB, and XY is perpendicular to PX.

7. Make accurate drawings to illustrate the elements of the following sets:

(i) $\{\mathrm{C} \mid \mathrm{C}$ is a circle, with its centre on a fixed circle, and which passes through a fixed point on the circumference of the fixed circle$\}$.

(ii) $\{\mathrm{C} \mid \mathrm{C}$ is a circle with its centre on a fixed circle, and which always passes through a fixed point outside the circle$\}$.

Another useful symbol is $\in$, which stands for "belongs to" and usually shows that an element belongs to a certain set. Thus $p \in A$ means that the element p belongs to the set denoted by A.

There are two special sets which often appear. The first is called the *universal set* and will be denoted by $\mathscr{E}$. This is the set of all things which could possibly be members of the sets in which we are interested. Thus for A, D, and F of Exercise 2, Question 1, the universal set could be all whole numbers, whereas if we wanted to include E as well, it would have to be enlarged to include all letters. The universal set could be defined as the set of all possible elements, but this would include a very large number of elements which are never likely to be considered in the context of the sets in which we are interested at any one time. We therefore restrict the scope of the universal set. The above definition is too vague to define a set properly, so the set rule for the universal set must be defined in detail as part of the discussion of the sets we are considering.

EXERCISE 3

1. Suggest suitable universal sets for each of the sets in Exercise 1, also Exercise 2, questions 1 and 2.

2. A is the set $\{e, g, f, h, i\}$, B is the set $\{1, 3, a, b, g, f\}$, and C is the set $\{x \mid x$ is either a whole number < 9 or one of the first 9 letters of the alphabet$\}$.
Which of the following statements are true?
$a \in A$; $a \in C$; $1 \in A$; $f \in A$; $b \in C$; $11 \in C$; $p \in B$.

3. In question 2 y is an element such that $y \in A$ and $y \in B$. What are all the possible elements which suit these conditions?

4. Repeat question 3 where the conditions are:
- (i) $y \in B$ and $y \in C$.
- (ii) $y \in B$ or $y \in C$.
- (iii) $y \in A$ and $y \in B$ and $y \in C$.
- (iv) $y \in A$ or $y \in B$ or $y \in C$.

5. Which if any of the following statements are true?
- (i) If $a \in A$, and $\mathscr{E}$ is the universal set then $a \in \mathscr{E}$.
- (ii) If $\mathscr{E}$ is the universal set, and $a \in \mathscr{E}$ then $a \in A$.

The other special set is called the *empty set*, and it is the set which has no element in it. It appears under disguises. The last example of Exercise 2, Question 1, is an empty set; another example would be A is $\{x \mid x$ is a negative whole number $> 20\}$ or B is $\{x \mid x$ is the number of pupils over twenty years old in the school$\}$, for both of these sets have no element which can satisfy the set rule. The empty set is usually denoted by the symbol $\emptyset$. (Note that it is not the set $\{0\}$ which has the one element 0 in it.)

As soon as we introduce letters for sets the question arises, "Can we make equations such as $A = B$ (where A and B are sets) and if so what do they mean?" In fact this statement is perfectly in order and means that the two sets A and B are the same set. Thus A could be $\{a, b, c, d\}$ and B could be $\{d, a, c, b\}$ or A could be $\{1, 2, 3, 0\}$ and B $\{x \mid x$ is a whole number $-1 < x < 4\}$.

Definition 3. Two sets are said to be equal if they consist of the same elements. Notice that the sets $\{a, b, c\}$ and $\{a, c, a, b, a, c\}$ are equal sets, as the number of times one element is listed does not matter.

So far we have not said anything about the size of the sets which we are considering, and the sets which have appeared have had a certain limited number of elements. This number of elements can be considered as a measure of the size of the set.

Definition 4. The *cardinal number* of a set is the number of elements in that set.

The cardinal number of a set A is written as $n(A)$. For example if $A = \{a, b, d, g, h\}$ then $n(A) = 5$. Most of the sets which we have considered so far have a finite number of elements but this is not necessarily so. For example the set $X = \{x \mid x > 1\}$ has an infinite number of elements. Such a set is said to be infinite, or its cardinal number to be infinitely large. We shall be interested mainly in the cardinal numbers of finite sets.

Note that we may not say that "the cardinal number equals infinity", because infinity is a concept and not a number. The concept is "a quantity larger than anything which I have yet considered". The reason why this concept may not be treated like a number is that with this definition of infinity it does not obey the ordinary rules for numbers. To illustrate this let us suppose that infinity is a number (represented by I) and that x is any other number, then the following equations must all be true:

$$I+I = I,\ I\times I = I,\ I+x = I,\ I\times x = I,\ I\div x = I,\ x\div I = 0$$

and the expression $I\div I$ can take any value according to how we arrive at each of the infinities. These equations show that I does not behave as any other (finite) number, and so special care must be taken when talking about operations involving infinite quantities. Just to treat infinity as any other number will clearly lead to unsuspected errors.

EXERCISE 4

1. Find the cardinal numbers of the following sets. If they are not finite, say that the cardinal number is infinitely large.

$\{x \mid x \text{ is a whole number where } 0 < x < 27\}$

$\{x \mid x \text{ is any number } -1 < x < 1\}$

$\{x \mid x \text{ is a factor of } 36\}$

$\{x \mid 2^x < 16\}$

2. Find the cardinal numbers of the following sets:

$\{a, b, 1, 5, x, z, t, p\}$

{All the circles through three non-collinear fixed points}

{All the straight lines which can be drawn through three non-collinear fixed points}

3. x is an element such that $x \in A$, and also $x \in B$. If X is the set of all such x, is it possible that $X = A$? Give an example to illustrate your answer.

4. x is an element such that either $x \in A$ or $x \in B$. If X is the set of all such x, is it possible that $X = A$? Give an example to illustrate your answer.

All that we have done so far probably sounds very abstract, and most theoretical ideas are much easier to understand if we can draw some kind of diagram to illustrate what we are talking about. The method of illustrating sets is to draw very simple diagrams called *Venn diagrams*. These are named after John Venn, who was a Cambridge philosopher and lived from 1834 until 1923. He was mainly interested in logic, a subject which is closely allied to mathematics. All we have to do is to draw an oval or circle (or any enclosed shape), the inside of which represents all the elements which belong to the set which we are illustrating. If we want to illustrate two sets, we draw two ovals, one for each set, and so on. If we start drawing pairs of ovals on a piece of paper, we will find that there is only a limited number of different ways in which this can be done. When we think that we have found all the possibilities (and not before) we can proceed.

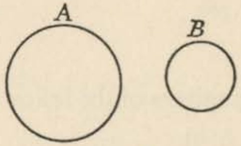


FIG. 1

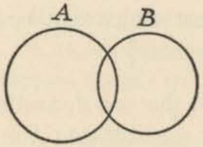


FIG. 2

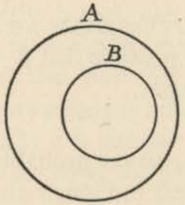


FIG. 3

We should have found only three different possibilities for the diagrams and these are shown in figures 1, 2, and 3. Each diagram illustrates a different kind of relationship between the two sets A and B. In figure 1 the sets A and B are quite separate (i.e. the elements of A are quite different from those of B); for example, A could be the set $\{1, 2, 3\}$ and B the set $\{a, b, c\}$. Such sets are said to be *disjoint*. On the other hand figure 1 could not represent the sets $A = \{1, 2, 3, 4\}$ and $B = \{4, 5, 6, 7\}$ as these sets have the element 4 in common, and so their Venn diagram must have a part where the circles of A and B overlap. In figure 2 this is precisely what happens, and there is an area which is common to the circles representing both the set A and the set B. Another example which would require this diagram could be $A = \{1, 2, 3, 4, 5, 6\}$ and $B = \{1, 2, 5, 7, 11\}$ and the diagram could then be labelled more fully as in figure 4.

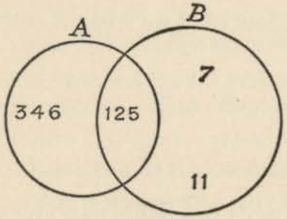


FIG. 4

In the case of figure 3 all the elements of B are also elements of A, and B is said to be completely contained in A; alternatively B is a subset of A (meaning that it is a set in its own right, but also that all its elements are already elements of another set A). An example of this case would be for A to be $\{1, 2, 3, 4, 5, 6\}$ and B to be $\{2, 4, 6\}$ and then the diagram could be more fully detailed as in figure 5.

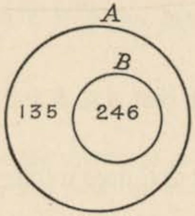


FIG. 5

This last case where B is contained in A is usually written symbolically as $B \subseteq A$.

Definition 5. B is a subset of A if B is a set and all of its elements are also elements of A. Notice that this definition of a subset allows any set to be considered as a subset of itself. If we want to exclude this case we talk of a *proper subset*.

Definition 6. A proper subset B of a set A is a subset where A has elements which are not in B.

If B is a proper subset of A we write $B \subset A$.

EXERCISE 5

1. Is the empty set $\emptyset$ a subset of any set A?

2. If $n(B) = p$ and $n(A) = q$, what can you say about p and q if (i) $B \subseteq A$ and (ii) $B \subset A$.

3. If $n(A)$ and $n(B)$ are both infinite, is it possible to have (i) $A \subset B$, (ii) $B \subset A$, (iii) $A \subseteq B$, and (iv) $B \subseteq A$?

4. If $n(A)$ is finite but $n(B)$ is infinite, are the four examples of question 3 possible?

5. A is the set $\{a, b, c, d\}$. Write down all the subsets of A.

6. $X = \{x \mid x \in A \text{ and also } x \in B\}$, is $X \subseteq B$? Is it possible to have $X = A$?

7. If we can show both that $A \subseteq B$ and also that $B \subseteq A$, is it true that $A = B$?

8. If $A \subseteq B$ and also $B \subseteq C$ does it follow that $A \subseteq C$?

9. If $A \subseteq B$ and also $A \subseteq C$, does it follow that $B \subseteq C$?

10. $A = \{\text{all circles through two fixed points}\}$, and B is the subset of circles which also have a radius of 1 inch. What is the value of $n(B)$? Does this answer depend on the distance apart of the two fixed points?

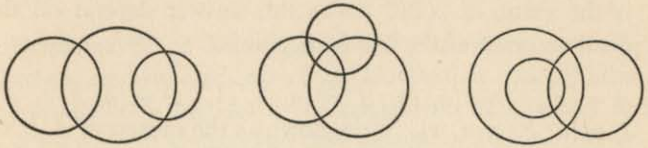
11. $A = \{a\}$, $B = \{a, b)$, $C = \{a, b, c\}$, $D = \{a, b, c, d\}$, $E = \{a, b, c, d, e\}$. Write down all the subsets of A, B, C, and D. How many are there in each case? Can you predict the number of subsets there will be in the set E? If P is a set where $n(P) = r$, how many subsets of P are there?

The case of figure 2 is usually taken as the standard way of drawing a Venn diagram for two sets, if their actual detailed composition is not known. The other two diagrams are really special cases of figure 2. Figure 1 is the case where the common part of the two sets is the empty set, and figure 3 is the case where that part of B not in A has no element in it (i.e. is the empty set). Consequently if we use figure 2 when in fact we should be using one of the other two diagrams, then some part of the diagram would have to be labelled as the empty set at some stage of the argument. Notice that although we can say that figures 1 and 3 are special cases of figure 2, we cannot say that figure 2 is a special case of either figure 1 or figure 3. As these were the only possible three figures for the Venn diagram of the two sets, figure 2 is described as the general Venn diagram for two sets.

EXERCISE 6

1. Select sets from the universal set consisting of all members of the school which have Venn diagrams as in figures 1, 2, and 3.

2. From the universal set of all human beings find subsets A, B, and C which fit the three Venn diagrams given in figure 6.



(a) (b) (c)

FIG. 6

3. Find sets E, F, and G from the universal set of all members of the school which fit the Venn diagrams of question 2.

4. Draw Venn diagrams to illustrate the following sets, labelling each part of the diagram with the elements which go with it. $A = \{p, r, s, t, v\}$ $B = \{a, b, t, v\}$ $C = \{x, y, v, t\}$

5. Which is the most general of the three diagrams of question 2?

These Venn diagrams immediately give a clear way of seeing how the elements of one set are related to those of another. The first of these we can see most clearly in figure 2, and it is that part which is common to the two sets. We can further illustrate it by shading the Venn diagram of the two sets (figure 7).

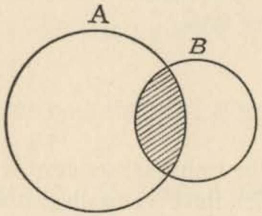


FIG. 7

Definition 7. The *intersection* of two sets is the set of all elements which are common to both sets.

The intersection of sets A and B is written symbolically as $A \cap B$ and is read as "A intersection B".*

Notice that although the intersection of the sets was clearly apparent in figure 2 the above definition holds in all cases. In figure 1 there are no elements common to both A and B, so we have that $A \cap B = \emptyset$. In figure 3 all elements of B are also common to A and so we have $A \cap B = B$.

EXERCISE 7

Describe the sets $A \cap B$ for each of the following pairs of sets A and B below:

	A	B
1.	$\{a, b, s, d, g, h, j, l, m\}$	$\{s, m, j, d, i, p, r, f, l, a, x\}$
2.	$\{x \mid x$ all prime numbers $< 20\}$	$\{x \mid x$ all odd numbers $< 20\}$
3.	$\{x \mid x$ all possible whole number factors of $30\}$	$\{x \mid x$ all possible whole number factors of $24\}$
4.	$\{y \mid y$ is a soldier$\}$	$\{y \mid y$ is an officer$\}$
5.	{all members of the sixth form}	{all boys in the school under 14 years old}
6.	{all five-sided plane figures}	{squares, equilateral triangles, rectangles}
7.	{parallelograms}	{rectangles}
8.	$\{x \mid x$ is a multiple of $3\}$	$\{x \mid x$ is a multiple of $5\}$

---

* Sometimes as A "cap" B.

The other method of combining sets is as follows:

Definition 8. The *union* of two sets A and B is the set of all those elements which are members of either A or B, or both A and B.

Using figures 1, 2, and 3 again, the union of the two sets A and B is the shaded area in each case (figure 8).

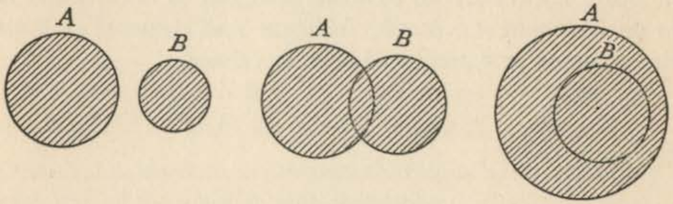


FIG. 8

The symbolic way of writing the union of two sets is $A \cup B$ and it is read as "A union B". Note that in the case of figure 3 $A \cup B = A$.

EXERCISE 8

1. Describe the sets $A \cup B$ for each of the pairs of sets A and B which appear in Exercise 7.
2. If $A \cup B = A$ is it true that (i) $A \subseteq B$ or (ii) $B \subseteq A$? Give an example to illustrate your answer.
3. If $A \cap B = A$ is it true that (i) $A \subseteq B$ or (ii) $B \subseteq A$? Give an example to illustrate your answer.
4. Is it always true that $A \cap B A \subseteq$? Is it true if we write $A \cup B \subseteq A$?
5. $X = \{x \mid x \in A \text{ or } x \in B \text{ or } x \in \text{ both } A \text{ and } B\}$. Write down the connection between X, A, and B.

6. $X = \{x \mid x \in A \text{ and } x \in B\}$. Write down the connection between X, A, and B.
7. $A = \{a, b, c\}$. The subsets of A are A, $X = \{b, c\}$, $Y = \{c, a\}$, $Z = \{a, b\}$, $U = \{a\}$, $V = \{b\}$, $W = \{c\}$, and $\emptyset$. If P is any one of these subsets and Q any other, show that $P \cup Q$, and $P \cap Q$ are both subsets of A.
8. Using the data of question 7 make out the two tables of figure 9—one to show the set $P \cup Q$ for any two sets P and Q, and the other to show $P \cap Q$.

∪	A	X	Y	Z	U	V	W	∅
A								
X								
Y								
Z								
U								
V								
W								
∅								

∩	A	X	Y	Z	U	V	W	∅
A								
X								
Y								
Z								
U								
V								
W								
∅								

FIG. 9

Look at each table. Is it possible to find one set in each which behaves in the same way as 0 does in the addition of ordinary numbers?

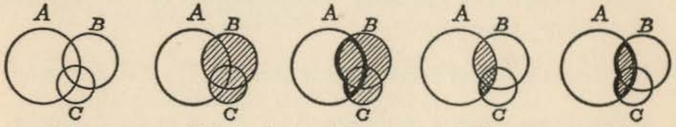
9. Is it true that (i) $A \cup B = B \cup A$ and (ii) $A \cap B = B \cap A$?
10. Write down what is represented by $A \cap B$ in the following cases:
    (i) A = all points inside a circle.
    B = all points on a line through the centre of the circle.
    (ii) A = all points on the surface of a sphere.
    B = all points lying in a plane which passes through the centre of the sphere.

(iii) $A =$ all the points lying on the surface of a cylinder.
$B =$ all the points lying in a plane which intersects the cylinder.

(iv) $A =$ all the points lying on the surface of a cone.
$B =$ all the points lying in a plane which intersects the cone.

The relative value of Venn diagrams soon becomes apparent when more complicated combinations of sets have to be considered. For example suppose that we wish to illustrate that $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$, where the equals sign means, as previously, "the same set". The first thing to note is the significance of the brackets, which are used in the same way as in algebra or arithmetic—they indicate which part of the combination should be worked out first. Thus on the left-hand side of the equals sign we must do $B \cup C$ first and then consider its intersection with the set A. By shading a Venn diagram we can illustrate this process as follows.

We will start with the set diagram (figure 10(*a*)) and first shade $B \cup C$ which gives (*b*). The intersection with the set A is shown by thickening in the outline of the area involved (*c*).



(a) (b) (c) (d) (e)

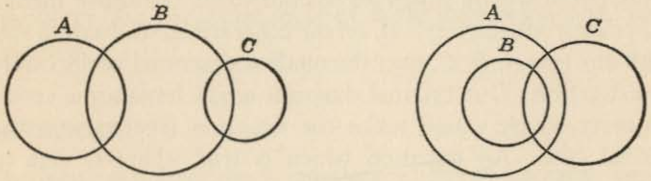
FIG. 10

If we now consider the other side of the equals sign, we must first shade $A \cap B$ and $A \cap C$ (*d*). Next we must consider the union of these two shaded areas, and again we thicken in the outline of this union (*e*).

Clearly the part of figure 10 which is outlined in each case is the same, and this would indicate that the two sides of the equation are in fact the same set, and so the equation is true—not just for three special sets, A, B, and C, but for all sets A, B, and C which fit the Venn diagram we have used.

EXERCISE 9

1. Repeat the same process to show that this equation is also true for three sets represented by figure 11 (*a*) and (*b*).



(a) (b)

FIG. 11

2. Using figure 11*a* for the three sets A, B, and C, find whether the following equations are true or not:

(i) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
(ii) $A \cap (B \cap C) = (A \cap B) \cap C$
(iii) $(A \cup B) \cap C = A \cup (B \cap C)$
(iv) $A \cup (B \cup C) = (A \cup B) \cup C$

3. Write x for A, y for B and z for C, $+$ for $\cup$ and $\times$ for $\cap$ (where x, y, and z are ordinary numbers) in the expressions $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ and $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$. You now have some ordinary equations involving algebra—are they correct? If not, compare the differences in the processes of taking unions and intersections of sets with the ordinary process of multiplication and addition of algebra. Is the situation improved if we write + for $\cap$ and $\times$ for $\cup$?

The above method of "proving" true equations involving combinations of sets is very convenient and straightforward, but it can be dependent on the relation implied between the sets A, B, C, etc., in the choice of the original Venn diagram, on which the rest of the argument is based. If we know something of the sets on which we wish to work, the above method is perfectly satisfactory. If, on the other hand, we have to start with any sets A, B, C, etc., the method described needs careful investigation. Our original diagram might have some special property which would make the equation true only in that special case. An equation which is true whatever sets (or numbers if it is an ordinary algebraic equation) are substituted for the letters, and not just in special circumstances, is called an *identity*. In order to prove an identity using a Venn diagram we must have the most general Venn diagram when we start.

To show how things can go wrong we will consider an example from ordinary algebra. Suppose that we wish to show that $2(x+1)-5$ simplifies to $x+7$. We take a special value of x (instead of any value at random) and say "try $x = 10$; if it works it must be true". Well, we should find that we get $2(10+1)-5 = 22-5 = 17$, and also that $10+7 = 17$, and so the expression would appear to be correct. On the other hand simple algebra, working out the bracket and simplifying, quickly shows that in general this is not true. (Why should $x = 10$ make it appear to be true? Are there any other values of x which would also do so?)

An alternative method of proving these identities, although rather more difficult to apply, gets over the difficulty of possibly having proved a special case. The method looks at what each side of the expression means as far as any single element is concerned, and shows that the two sides reduce to the same result in the end. Consider again the expression $A \cap (B \cup C)$. An element which belongs to the set defined by this expression must belong to A and also to at least one of B or C, which is the same as belonging to (A and B) or (A and C), i.e. to the set defined by $(A \cap B) \cup (A \cap C)$. Consequently the expression must be true, and since we have not used any special property of a particular set A or B, the expression is in fact an identity.

EXERCISE 10

1. Write out Exercise 9, question 2, using the same kind of argument.

2. Which of the following "equations" are in fact identities, i.e. hold true whatever numbers are substituted for the letters?

(*a*) $3x+5 = 7$ (*b*) $2x+7 = 2(x+1)+5$

(*c*) $x^2+4x-1 = (x+2)^2-5$ (*d*) $2(x+1)-3(x-2) = x-5$

(*e*) $\dfrac{2}{x} = \dfrac{3}{x+1}$ (*f*) $\dfrac{2}{x} = \dfrac{2x^2+2}{x^3+x}$

3. Which of the following equations are identities (where the letters stand for sets):

(*a*) $A \cap B = B$

(*b*) $A \cup (B \cup C) = (A \cup B) \cup C$

(*c*) $A \cap (B \cup C) = (A \cap B) \cup C$

(*d*) $A \cup B = A$

(*e*) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

(*f*) $A \cup B = B \cup A$

4. Which of the parts of question 3 are statements which are never true?

We have already considered what we mean by a pair of equal sets—that they are in fact the same set. There is another relation between sets which is important. Take a piece of paper and draw a Venn diagram to illustrate the connection between the pair of sets $\{a, b, c, d\}$ and $\{a, f, g\}$ and then another diagram to show the relationship between the pair $\{1, 2\ \ 3, 4\}$ and $\{1, 7, 8\}$. You should get the same diagram in each case. Consequently any property which can be established for the first pair will automatically be true for the other pair, with an appropriate exchange of elements of one pair to those of the other. Such systems of sets are called *equivalent*, and their basic property is that for every element in the first system there can be found one element in the second which can be paired with it. Thus the basic structure of the two systems is identical, although the elements are different. $1 \rightarrow a$, $2 \rightarrow b$, $3 \rightarrow c$, $4 \rightarrow d$, $7 \rightarrow f$, and $8 \rightarrow g$ would be one such pairing of elements. Note that there can be more than one possible way of pairing off elements, for, providing that $1 \rightarrow a$, the other numbers and letters can be paired as we wish, the basic structure of the two pairs of sets will still be maintained. Note also that it does not matter which way round we think of the pairing being done. We would get precisely the same result if we wrote it as $a \rightarrow 1$, $b \rightarrow 2$, $c \rightarrow 3$, etc. For this reason it is usually written as $a \leftrightarrow 1$, $b \leftrightarrow 2$, etc. and it is called a "one-to-one mapping" of the elements of one system on to those of another. (One-to-one is often written as 1–1.)

Definition 9. A one-to-one mapping of the elements of one set on to another is an arrangement whereby an element of one set is paired with one and only one element of the other set, and all elements of each set belong to such a pairing.

As another illustration think of the titles of the elements being written on discs, so that an element of the first set is on one side, and its pair is written on the other. Then if this is to be a 1–1 mapping, all discs must have an element on each side; and if an element is seen on a disc, then there can only be one possible element on the other side of the disc. An element can be paired with only one other element, and must always be paired with the same element every time it appears.

Definition 10. Two systems of sets are said to be equivalent when there exists at least one 1-1 mapping of the elements of one system on to those of the other.

Notice that any two sets which are equal must automatically be equivalent, for the mapping which puts each element into itself will be a 1–1 mapping. On the other hand it is clear, from the example of equivalent sets opposite, that sets which are equivalent may not necessarily be equal. Thus equivalence is a more general property than equality. It maintains the basic structure of the two systems without requiring that they should in fact be identical. Thus any property of one system which is not dependent on the actual details of the elements which make up the set, will also be a property of any equivalent set. In particular all equivalent systems will have the same Venn diagram.

EXERCISE 11

1. Show that if the two systems of sets consist of just one set each, then they are equivalent if the cardinal numbers of the sets are the same.

2. Write down a pairing of the elements which is a 1–1 mapping of the system $\{a, b, c, d, e\}\{a, e, f, g\}$ on to the system $\{1, 2, 5, 7, 9\}\{1, 11, 13, 7\}$.

3. Are the following systems of sets equivalent?

(*a*) $\{1, 3, 7, 5, 9\}\{1, 2, 3, 4\}$ and $\{p, a, z, x, t\}\{p, a, b, c\}$
(*b*) $\{a, b, p, z\}\{x, y, t\}$ and $\{1, 2, 3, 4\}\{4, 5, 6\}$
(*c*) $\{x \mid x \text{ is a factor of } 12\}$ and $\{x \mid x \text{ is a factor of } 20\}$
$\{x \mid x \text{ is a factor of } 15\}$ $\{x \mid x \text{ is a factor of } 9\}$
(*d*) {squares, rectangles} and $\{a, b\}$
{parallelograms} {all letters of the alphabet}

4. A = {all points on the circumference of a circle}.
Does the following construction give a 1–1 mapping of A on to a straight line in the same plane as A?
Take any point P in the plane of the circle and any point X which is on the circumference of the circle. Join PX and produce it (if necessary) to cut the line at Y. Keep P fixed and consider all possible points X. The mapping pairs X with Y.

5. Does the following construction give a 1–1 mapping of the set A of question 4 on to itself?
Take any point P in the plane of the circle, but not on the circle, and any X as before. Join PX and produce it (if necessary) to cut the circle again at Y. The mapping pairs X with Y. What happens if P lies on the circle?

Consider now a universal set consisting of all the pupils in the school, a set (which is of course a subset of the universal set) of all pupils who are in your mathematics set, and then a third set defined in a negative way—that is, all pupils in the school who are not in your mathematics set. This is the universal set with the original set removed from it, and is called the *complement* of the original set. Note that the complement is defined only when the set whose complement we want and the universal set are both clearly defined.

Definition 11. The *complement* of a set is the set of all elements which belong to the universal set but do not belong to the original set.

The complement of A is written as A'. It will be seen that $(A')' = A$, because an element which is in the universal set, but not in that part of the universal set "outside" A, must be "inside" A, i.e. is a member of A. Figure 12 shows the complement of A as the shaded portion. (It is usual to show the universal set as a rectangle rather than an oval, to stress that it is rather different from the other sets under consideration.)

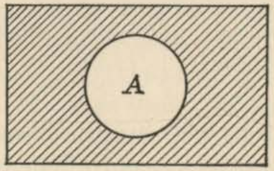


FIG. 12

EXERCISE 12

1. What are the complements of (i) the universal set, and (ii) the empty set? Notice that for these special cases it is not necessary to detail the universal set.

2. Write down the complements of the following sets for the given universal sets:

	Set	*Universal set*
(i)	$\{a, b, h, k, l, p, r, s, t, f\}$	{the first twenty letters of the alphabet}
(ii)	{parallelograms}	{four-sided figures with at least one pair of sides parallel}
(iii)	{all odd whole numbers}	{all whole numbers}
(iv)	$\{x \mid x < 3\}$	$\{x \mid x < 5\}$
(v)	$\{0\}$	{all numbers}

3. Show by means of shading Venn diagrams whether the following statements are correct:

(i) $(A \cup B)' = A' \cap B'$, (ii) $(A \cap B)' = A' \cup B'$, (iii) $A \cup B' = (A' \cap B)'$, and (iv) complete the bracket for the identity $A \cap B' = (A' \quad)'$, and then prove the result which you have predicted.

4. Prove the results of question 3 using the method of Exercise 10, question 1.

We will now consider what happens to the cardinal numbers of sets when we perform the operations of union and intersection on them. First an example, let $A = \{a, b, d, f\}$ and $B = \{a, c, g, h, j\}$. The $n(A) = 4$ and $n(B) = 5$, $A \cap B = \{a\}$ and $A \cup B = \{a, b, d, f, g, h, j, c\}$, and so $n(A \cap B) = 1$ and $n(A \cup B) = 8$. The question is whether it is possible to find any general rules about the sizes of $n(A \cap B)$ and $n(A \cup B)$ when we know the cardinal numbers of the two sets A and B.

We will consider the case of the intersection of two sets. The sets can be illustrated in a Venn diagram in three possible ways, and we shall assume that B is the set with the larger cardinal number. These three ways are illustrated in figures 13, 14, and 15.

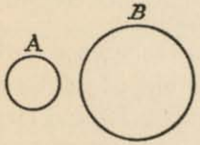


FIG. 13

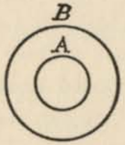


FIG. 14

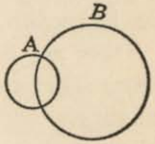


FIG. 15

In figure 13 the two sets have no element in common and we get $A \cap B = \emptyset$ so that $n(A \cap B) = n(\emptyset) = 0$. This clearly represents the smallest possible answer. In figure 14, $A \cap B = A$ and so $n(A \cap B) = n(A)$, or the answer then is the cardinal number of the smaller of the two sets. Again clearly this is the largest value which $n(A \cap B)$ can possibly take, as $(A \cap B)$ cannot consist of more elements than there are in the smaller of the two sets A and B; otherwise some elements of the intersection could not be included in the smaller set, which clashes with the definition of intersection. So far therefore we seem to have that $0 \leqq n(A \cap B)$ and $n(A \cap B) \leqq n(A)$, and we would expect that the other case of figure 15 would come to a value between these two values.

In figure 15 only part of A is included in B and so $n(A \cap B)$ must be less than it was when all of A was included; but the diagram would indicate that there may be some elements of A in B, and so $n(A \cap B)$ will be greater than the 0 of the case of figure 13. Taking all these results together, and putting $n(A) = p$, $n(B) = q$, and assuming that $p < q$, we have that $0 < n(A \cap B) < p$.

EXERCISE 13

Using the same kind of argument, show that

$$q < n(A \cup B) < p+q$$

The cardinal numbers of sets, together with Venn diagrams can be used to solve the following kind of problem:

There are 100 girls at a certain school in the science sixth form. They are all supposed to do at least one of the three languages, French (F), German (G), or Russian (R), but may do more than one if they wish. The senior language mistress produced the following breakdown to show what they are doing. A total of 45 do French, 30 German, and 25 Russian. Of these, 15 do German and French, 10 do German and Russian, and 12 French and Russian, and of these 8 do all three languages.

The headmistress said she wanted to see all those girls who are not doing any language at all. How many are there?

At first sight it would appear that $45+30+25 = 100$ do at least one language, and so the answer is nil, but a closer look will show that this is false. We will use a Venn diagram (figure 16), and label the sets of those who do French (F), etc.

The first thing to note is that 8 girls do all three languages, i.e. $n(F \cap G \cap R) = 8$. So we mark in that part of the diagram which represents $F \cap G \cap R$ its cardinal number (figure 17).

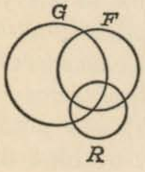


FIG. 16

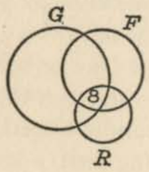


FIG. 17

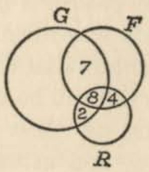


FIG. 18

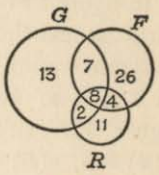


FIG. 19

Now 15 do German and French, but eight of these have already appeared, so the other part of $G \cap F$ must contain 7 girls, and this is then marked in (figure 18). Similarly for $F \cap R$ and $R \cap G$.

Finally a total of 30 do German, but 17 ($= 7+8+2$) have already been accounted for, and so there are only 13 left who do just the one language, German. Similarly there are 26 who do just French and 11 who do just Russian. Figure 19 can then be completed.

Now if we add up the total number of pupils we get $13+26+11+2+7+4+8 = 71$, so there are in fact 29 girls who have not arranged to do any language.

EXERCISE 14

1. At a school of 400 boys a total of 112 played rugby, 120 played hockey, and 168 played soccer. Of these, 32 played both soccer and hockey, 40 played rugby and soccer, and 20 played rugby and hockey; and of these, 12 played all three games. How many did not play any game, and how many played only one game?

2. A consumer research unit sent out a man to ask people whether they used (i) a ball-point pen, (ii) a fountain pen, and (iii) a pencil. He was told to interview 100 people, and his report read as follows:

Number of people using ball-point pens	50
Number of people using pens	23
Number of people using pencils	27

of these

20 used both a pen and pencil.
18 used both a ball-point and a pen.
10 used a pencil and a ball-point.

of these

5 used all three

The manager of the consumer research unit sacked the man, saying that he had produced a bogus survey. Why?

3. At a social club members are expected to take part in at least one activity—choral society, political discussion group, or theatricals. From a group of 25 people, 11 join the choral society; 7 of these also act, and of these 7, 2 do all three activities. There is a total of 12 politicians, only 2 of whom take part in just one activity. What are the maximum and minimum numbers of people who can be expected to appear for a rehearsal arranged by the theatrical society?

* * * * * * * *

Finally we might ask, "Is this just a peculiar sideline of mathematics, or are there deeper reasons for the ideas in this pamphlet?" In order to answer this question, we should look at the following sets:

$\{x \mid x^2+3x-4=0\}$

$\{x \mid x>1\}$

$\{(x,y) \mid 2x+3y=7\}$

{All points lying in a plane that are equidistant from a fixed point.}

{All lines lying in a plane that are equidistant from a fixed line.}

{All the pupils taking ordinary-level mathematics this year in the school who get the pass grade or higher.}

{All the trains that go non-stop from London to Brighton.}

We could go on listing sets, and finding that most of the objects and ideas around us fall naturally into sets. In mathematics a graph, or a locus of points, or an envelope of lines, or a collection of numbers with a certain property, all naturally form sets; what we have tried to do is to consider some of the formal developments which come from this simple basic fact. Since so many of the mathematical ideas that we come across are based on sets of this kind, we find that the language of set theory keeps reappearing. It is not therefore the detailed results of this pamphlet, but the approach, which is of most value—for here we have, in simple form, the very essence of mathematics. We start with a simple definition, in order that we shall know accurately what we are talking about—and moreover a definition of something which cannot be deduced from simpler ideas, a thing which mathematicians call an *axiom*—and from this we have built up a theory, making extra definitions when an obvious new development required them, so that what has emerged is an "algebra of sets", or the theory of how sets are interrelated.

If we want to see more of how an algebra of this kind may be developed into practical channels, we should look for a book on *Boolean Algebra* and in particular the applications to the circuitry of computers. If, on the other hand, we are more interested in the logical way in which a subject can be developed from basic axioms—using the same kind of language that we have used here—then we must find a book on logic. Mathematics is one kind of study of logic. Both these suggestions involve complications which are probably fairly difficult to understand, but they help to show how the ideas of this pamphlet represent a basis on which mathematics and other allied subjects may build.

SETS

Symbols

$>$	greater than
$\geqq$	greater than, or equal to
$<$	less than
$\leqq$	less than, or equal to
$\{\ \}$	the set consisting of the elements detailed inside the brackets
$\cup$	union
$\cap$	intersection
A'	complement of the set A
$\in$	belongs to
$\subseteq$	is a subset of
$\subset$	is a proper subset of
$\emptyset$	empty set
$\mathscr{E}$	universal set
$\mid$	such that.

Answers

EXERCISE 1

(*a*) Yes (*b*) Yes (*c*) No (*d*) Yes (*e*) Yes (*f*) No

EXERCISE 2

1. $A = \{1, 2, 3, 4, 5, 6, 7, 8\}$ $C = \{\frac{-4}{3}\}$, $D = \{-3, +3\}$
$E = \{pqr, prq, qpr, qrp, rpq, rqp\}$
F has no elements

2. $A = \{x \mid x \text{ is a whole number } > 17\}$
$B = \{x \mid x \text{ is a pupil who } < 15 \text{ years old}\}$
$C = \{x \mid x \text{ is a member of my form}\}$
$D = \{x \mid x \text{ is a factor of } 48\}$
$E = \{x \mid x \text{ is a member of my form who is over 20 years old}\}$
$F = \{x \mid x^2 + 1 = 0\}$

3. Both sets have no elements in them. [It is assumed that pupils at this level will understand "real numbers" for the term number.]

4. (i) Circle centre O.
(ii) The perpedicular bisector of AB.
(iii) A circle on AB as diameter.
(iv) The perpendicular bisector of AB.
(v) A circle with the fixed point as centre.

5. (i) A sphere centre O.
   (ii) A plane perpendicular to AB and through its mid-point.
   (iii) A sphere with AB as a diameter.
   (iv) A plane perpendicular to AB and through its mid-point.
   (v) A sphere with the fixed point as centre.

EXERCISE 3

1. From Ex. 1 (*a*) $\{x \mid x \text{ is a whole number}\}$
   (*b*) $\{x \mid x \text{ is a car with a British registration plate}\}$
   (*d*) $\{x \mid x \text{ is a pupil of the school}\}$
   (*e*) $\{x \mid x \text{ is a school}\}$

   From Ex. 2, Q. 1: (*a*) $\{x \mid x \text{ is a whole number}\}$
   (*b*) $\{x \mid x \text{ is a pupil}\}$
   (*c*) $\{x \mid x \text{ is any number}\}$
   (*d*) $\{x \mid x \text{ is a whole number}\}$
   (*e*) $\{x \mid x \text{ is any arrangement of any three letters}\}$
   (*f*) $\{x \mid x \text{ is any number}\}$
   Q. 2: (*a*) $\{x \mid x \text{ is a positive whole number}\}$
   (*b*) and (*c*) $\{x \mid x \text{ is pupil at the school}\}$
   (*d*) $\{x \mid x \text{ is a whole number}\}$
   (*e*) $\{x \mid x \text{ is a member of the school}\}$
   (*f*) $\{x \mid x \text{ is a number}\}$

   N.B. These answers are by no means unique.

2. (*a*) false (*b*) true (*c*) false (*d*) true (*e*) true (*f*) false (*g*) false

3. g, f

4. (i) 1, 3, a, b, f, g
   (ii) 1, 2, 3, 4, 5, 6, 7, 8, $a, b, c, d, e, f, g, h, i$
   (iii) f, g
   (iv) 1, 2, 3, 4, 5, 6, 7, 8, $a, b, c, d, e, f, g, h, i$

5. (i) true (ii) false

EXERCISE 4

1. (i) 26 (ii) infinitely large (iii) 9 (iv) infinitely large
2. (i) 8 (ii) 1 (iii) None
3. Yes
4. Yes

EXERCISE 5

1. Yes
2. (i) $p \leq q$ (ii) $p < q$
3. (i) Yes (ii) Yes (iii) Yes (iv) Yes
4. (i) Yes (ii) No (iii) Yes (iv) No
5. Notice that the subsets include $\emptyset$ and $\{a, b, c, d\}$
6. (i) Yes (ii) Yes
7. Yes 8. Yes 9. No
10. 2, 1, or 0 according as the two fixed points are less than, equal to, or more than 2 inches apart.
11. E has 32 subsets; general formula is 2^r where r is the number of elements in the set.

EXERCISE 6

5. (*b*)

EXERCISE 7

1. $\{a, s, d, j, l, m\}$

2. $\{1, 3, 5, 7, 11, 13, 17, 19\}$

3. $\{1, 2, 3, 6\}$

4. {army officers}

5. $\emptyset$

6. $\emptyset$

7. {rectangles}

8. $\{x \mid x$ is a multiple of $15\}$

EXERCISE 8

1. (i) $\{a, b, s, d, g, h, l, m, i, j, p, r, f, x\}$
  (ii) $\{x \mid x$ all odd numbers < 20 and $x = 2\}$
  (iii) $\{1, 2, 3, 4, 5, 6, 8, 10, 12, 15, 24, 30\}$
  (iv) {soldiers}
  (v) {all members of the sixth form and all boys under 14 years of age}
  (vi) {all squares, equilateral triangles, rectangles, and five-sided plane figures}
  (vii) {parallelograms}
  (viii) $\{x \mid x$ is a multiple of 3 or $5\}$

2. (i) No (ii) Yes

3. (i) Yes (ii) No

5. $X = A \cup B$ S 12115

6. $X = \mathrm{A} \cap B$ S 12114

9. (i) Yes (ii) Yes

10. (i) a diameter of the circle
  (ii) a great circle of the sphere
  (iii) an ellipse
  (vi) any one of the conic sections i.e. either an ellipse, or a circle, or a parabola, or an hyperbola or two straight lines.

EXERCISE 9

2. (i) True (ii) True (iii) False (iv) True.

3. (i) No (ii) No. Consider the equivalent of $a + b.c$ in each case.

EXERCISE 10

2. a, d, and e are not identities.

3. a, c, and d are not identities.

4. c.

EXERCISE 11

2. $a \rightarrow 1$, $e \rightarrow 7$, $b \rightarrow 2$, $c \rightarrow 5$, $d \rightarrow 9$, $f \rightarrow 11$, $g \rightarrow 13$

3. (*a*) yes (*b*) no (*c*) no (*d*) no

4. No

5. Yes

EXERCISE 12

1. (i) $\emptyset$ (ii) $\mathscr{E}$

2. (i) $\{c, d, e, g, i, j, m, n, o, q\}$
  (ii) {trapeziums}
  (iii) {all even whole numbers}
  (iv) $\{x \mid 3 \leq x < 5\}$
  (v) {all non-zero numbers}

EXERCISE 14

1. 252 play only one game, 80 play no games.

3. Maximum 19; minimum 4.